Stabilizing the absolutely or convectively unstable homogeneous solutions of reaction-convection-diffusion systems

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We study the problem of stabilization of a homogeneous solution in a two-variable reaction-convection-diffusion one-dimensional system with oscillatory kinetics, in which moving or stationary patterns emerge in the absence of control. We propose to use a formal spatially weighted feedback control to suppress patterns in an absolutely or convectively unstable system and pinning control for a convectively unstable system. The latter approach is very effective and may require only one actuator to adjust feed conditions. In the former approach, the positive diagonal elements of the appropriate dynamics matrix are shifted to the left-hand part of the complex plane to ensure linear (asymptotic) stability of the system according to Gershgorin criterion. Moreover, we construct a controller that (with many actuators) will approach the global stability of the solution, according to Liapunov's direct method. We apply two alternative approaches to reveal the unstable modes: an approximate one that is based on linear stability analysis of an unbounded system, and an exact one that uses a traditional eigenstructure analysis of bounded systems. The number of required actuators increases dramatically with system size and with the distance from the bifurcation point. The methodology is developed for a system with learning cubic kinetics and is tested on a more realistic cross-flow reactor model.

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I. INTRODUCTION

Control of spatiotemporal patterns in distributed dynamical systems is currently attracting significant attention. Several recent applications in reaction-diffusion and fluid dynamics systems derived the methodology for control of a desired spatiotemporal pattern ([1–6] and references therein). Our group has derived control procedures for reaction-convection-diffusion (RCD) systems. Control of spatiotemporal patterns in a distributed dynamical system is significantly more difficult than that of lumped models since it requires multiple sensors (or continuous imaging or spatial filtering of profiles [7]) and multiple spatially dependent actuators (see [8] for discussion) or even a spatiotemporally dependent actuator that can control traveling solutions. Distributed systems are described by an infinite-dimensional set of ordinary differential equations (ODE's) and we need to control the unstable modes without destabilizing other (stable) modes. The instability of spatiotemporal patterns is typically characterized by a small number of unstable modes, and various approaches to that end were suggested. The number of unstable modes, associated with the homogeneous solution, however, increases with system size and with the bifurcation parameter, and its stabilization is the subject of the present article.

In the present article, we study the methodology for stabilizing the homogeneous $(x=u=0)$ solution of the RCD one-dimensional system with oscillatory kinetics

$$\mathrm{Le}\; x_t + Vx_z - x_{zz} = f(x,u), \quad u_t + Vu_z - Du_{zz} = g(x,u), \tag{1a}$$

subject to the Danckwert's boundary conditions

$$x_z(0) = V[x(0) - x_0], \quad x_z(L) = 0; \tag{1b}$$

$$Du_z(0) = V[u(0) - u_0], \quad u_z(L) = 0. \tag{1c}$$

Here Le is the Lewis number, V, D are convective velocity and diffusion coefficients. For most of this work, we employ the following simple (oscillatory) polynomial kinetics

$$f(x,u) = -x^3 + x + u, \quad g(x,u) = -du - \beta x, \tag{1d}$$

while in Sec. VI, we employ a more realistic cross-flow reactor in which a first-order activated reaction occurs. We choose a model that incorporates convection since the interplay between absolute instability, due to autocatalytic kinetics, and convective instability, which can be induced by convection [9,10], makes the control problem especially interesting.

This model (and similar ones) has been employed for many years in simulating patterns in chemical systems. Most mechanisms employ an activator-inhibitor interaction with a sufficiently wide difference of their diffusive (i.e., $D \gg 1$), convective or capacity $(\mathrm{Le} \gg 1)$ properties. Eq. (1a)–(1d) accounts for two variables: the main autocatalytic variable (activator, x) which undergoes reaction, advection, and diffusion, and an inhibitor (u) which may be advected by the flow. With $V=0$ and when D is sufficiently large, this model forms the extensively investigated Turing pattern-forming mechanism [11,12]. With $\mathrm{Le} \gg 1$, it forms the differential capacity pattern mechanism. Our group showed that stationary-periodic or even-complex (or -chaotic solutions) can emerge when the activator capacity is sufficiently large $(\mathrm{Le} \gg 1)$, while both the inhibitor and activator flow at the same rate [9]; the inhibitor diffusivity is not crucial for the establishment of these patterns and it can be set to zero $(D=0)$ and we need to specify only the feed conditions, $u(0)$. Other groups have focused on different convective terms of x and u and termed it the differential flow induced chemical instability [13]. Patterns in RCD systems have been a subject of investigation in the past decade; a unifying view of the various recently proposed versions of this mechanism is presented in [9,10].

This model with convection and $\mathrm{Le} \gg 1$ was employed by us as a learning model for a certain class of chemical reactors, and specifically for modeling the cross-flow reactor in which the main reactant (u) is dispersed along the reactor, rather than fed at one port as in traditional chemical reactors [14]. The identity of the various variables is not crucial, but in order to build a reasonable control strategy, we need to consider the technological feasibility of the various control schemes. In our interpretation, x is the fluid temperature while u is the reactant concentration; other interpretations that refer to the Belousov-Zhabotinski reaction were also suggested.

To simplify the analysis to a single-variable presentation, assume that u is responding fast (i.e., $\mathrm{Le} \gg 1$) and its balance is assumed to be in steady state. For the case $D=0$, the original system (1) is reduced to one-variable integro-differential equation

$$\mathrm{Le}\ x_t + V x_z - x_{zz} = -x^3 + x + u(x), \tag{2a}$$

$$u(x(z,t)) = u(0)e^{-(d/V)z} - \frac{\beta}{V}\int_0^z e^{-(d/V)(z-\xi)}x(\xi)d\xi, \tag{2b}$$

$$x_z(0) = V[x(0) - x_0]; \quad x_z(L) = 0. \tag{2c}$$

While we have analyzed the control problem of fronts and patterns, in systems similar to Eq. (1), the control of the homogeneous state is a challenging problem since, upon crossing the stability threshold of the system, self-organized patterns of a certain characteristic wave number start to emerge and the number of such modes increases dramatically with changing the size of the system or with varying a bifurcation parameter. The stability threshold and these modes (wave numbers), along with their group velocity, can be easily determined for an unbounded system. This can serve as an asymptotic solution, but the behavior of a bounded system is different due to the boundary conditions. Specifically, we showed [10] that system (1) with $\mathrm{Le} \gg 1$ and $D=0$, admits traveling wave solutions upon crossing the threshold of the unbounded system, but in a bounded system with $x_o \neq 0$ there may exist another (higher) threshold, which can be determined by linear analysis, so that stationary patterns emerge sufficiently far from it. These result from the perturbations introduced by the boundary conditions $(x_o \neq 0)$. In the vicinity of this threshold we may still find traveling solutions, but their domain of stability cannot be determined analytically. We elaborate further on this structure later.

RCD systems may exhibit absolutely unstable or convectively unstable states: The system is said to be convectively unstable when perturbations grow in a frame moving with the flow. The system is absolutely unstable when perturbations grow in the physical frame. We propose to use formal spatially-weighted feedback control to suppress patterns in an absolutely or convectively unstable system and pinning control for a convectively unstable system. We consider two approaches for designing the feedback control laws. The first one, which is based on analysis of unbounded systems, approximates the eigenfunctions from the wave numbers of the unbounded system, and is more suitable for arbitrarily large systems. The second approach accounts for the boundary conditions and we use the expansion in terms of eigenfunctions of the linear operator of the partial differantial equations (PDEs). These are analyzed in Secs. II and III. We study the control design, using a truncated N-term spectral representation of the model, based on asymptotic stability analysis, using the Gershgorin theorem, as well as with a design based on Liapunov direct method.

We study two pinning strategies. (i) By controlling the desired solution $(x=0)$ at several set points (i.e., pinning), the actual system size to be studied is that between two such pins. The system stability is assured by placing many (preferably equally distant) pins. To determine these separations, we should search for the largest stable bounded system with Dirichlet boundary conditions. (ii) Controlling the feed conditions to $x(0)=0$ and $u(0)=0$ should stabilize the homogeneous solution in a short systems in a domain of absolute instability and in a long system in a domain of convective instability: The perturbations in the system will be washed out by the flow, if the velocity is sufficiently large. Within the domain of absolute instability and in a long system, the small perturbations of the feed conditions will be amplified. This seems to be simplest control strategy in the convective instability domain, if the system parameters are known exactly. The strategy of pinning control is analyzed in Secs. IV and V. Pinning control, in the form of periodically spaced actuators that apply local perturbation, has been employed in many studies of differential and difference equations (e.g., Auerbach [15] studied this approach in difference equation that produce chaotic solutions in the absence of control.) The actual implementation of such control in reacting systems is technically quite difficult.

The methodology is developed for a system with learning cubic kinetics and is tested on a more realistic cross-flow reactor model in Sec. VI. Specifically, we introduce a feedback control of feed conditions, for systems with certain uncertainty of the parameters.

Problem statement: Let us consider the control of the homogeneous state of the reduced model (2)

$$\begin{gathered}\mathrm{Le}\ x_t + V x_z - x_{zz} = -x^3 + x + u(x) + \lambda, \\ x_z(0) = V[x(0) - x_o], \quad x_z(L) = 0,\end{gathered} \tag{3}$$

$[u(0)=0]$, where $\lambda=\lambda(z,t)$ is the control variable, $|x_o| \ll 1$ and constant. In the absence of control, there exists a homogeneous solution ($x_s=u_s=0$ when $x_0=0$ or a solution that approaches this value far from feed when $x_o \neq 0$). This state becomes unstable within a domain of V and moving or stationary waves emerge. We will consider stabilizing control action that can be presented in the general form

$$\lambda(z,t) = -g\sum_{n=1}^{\eta} v_n(t)\psi_n(z), \tag{4}$$

where g is a gain coefficient, $v_n(t)$ are independent manipulated inputs, while $\psi_n(z)$ define the spatial distribution of the control. We consider two feedback control approaches with $\psi_n(z)$ that are either the exact or approximate eigenfunctions. We also consider a strategy with $\psi_n(z)=\delta(z-z_n)$ and $v_n \to \infty$, which implies pinning the solution in some points z_n, $z_n \in [0,L]$.

We realize that control of the form of Eq. (4) may be too restrictive, as the open system may acquire traveling waves solutions. In that case, we may try to control the system using an actuator that travels with the wave at the same velocity. We do not consider traveling wave control here for the following reasons: (i) traveling waves appear in unbounded or ring-shaped systems, and their domain of existence in bounded systems diminishes with decreasing the system size [10], (ii) implementing such control is typically technically difficult, and (iii) the simple Eq. (4) control is typically sufficient as the control itself stabilizes the front.

II. APPROXIMATE CONTROL FOR LONG SYSTEMS

In this section, we assume that the system length L is sufficiently large to approximate functions $\psi_n(z)$ by eigenfunctions of the unbounded system. In a bounded system, we use a discrete set of these functions that satisfies the boundary conditions with certain phase shift. Once these eigenfunctions are approximated for specified L and a bifurcation parameter (V), the control is applied to suppress unstable eigenfunctions in the form of

$$\lambda = -g\sum_{s=s_1}^{s_l} \langle \overline{x}(z,t), \psi_s(z)\rangle \psi_s(z), s_1 > 0, \tag{5}$$

where $\overline{x}=x-x_s$ and $\langle \cdot, \psi_s(z)\rangle$ denotes space-averaged.

We analyze first the bifurcation from the homogeneous solutions (x_s,u_s) that satisfy $f(x_s)+u_s=g(x_s,u_s)=0$. Linearizing Eq. (3) with $\lambda=0$, we obtain an equation for the evolution of $\overline{x}$

$$\mathrm{Le}\;\overline{x}_t + V\overline{x}_z - \overline{x}_{zz} = f_x(x_s,u_s)\overline{x} + u(\vec{x}). \tag{6}$$

Expanding $\overline{x}(z,t)$ as a superposition of harmonics and imposing component $x_k(z,t)$ of the form $x_k(z,t)=e^{\omega_k t}e^{ikz}$ with complex $\omega_k=\omega=\eta+\sigma i$ yields the dispersion relation

$$D(\omega,k,V) = \mathrm{Le}(\eta+\sigma i) + ikV + k^2 - \left.\frac{\partial f}{\partial x}\right|_{x=x_s} + \frac{\beta}{(d+ikV)}. \tag{7}$$

The neutral curve $(\mathrm{Re}\,\omega=\eta=0)$ is defined by the real and imaginary parts of the equation above. Noting that $\partial f/\partial x|_{x=x_s}=1$ $(x_0=0)$, we find

$$-\sigma\,\mathrm{Le}\;Vk - V^2k^2 + d(k^2-1) + \beta = 0,$$

$$\sigma\,\mathrm{Le}\;d + dVk + Vk(k^2-1) = 0,$$

defining the neutral curve in the $V(k)$ plane as

$$V^2 = \frac{-\beta d - d^2(k^2-1)}{k^2(k^2-1)}, \quad \sigma = -\frac{kV(d+k^2-1)}{\mathrm{Le}\;d}. \tag{8}$$

Two critical points may be defined on the neutral curve: its minimum (V_m,k_m) and the point of zero wave velocity

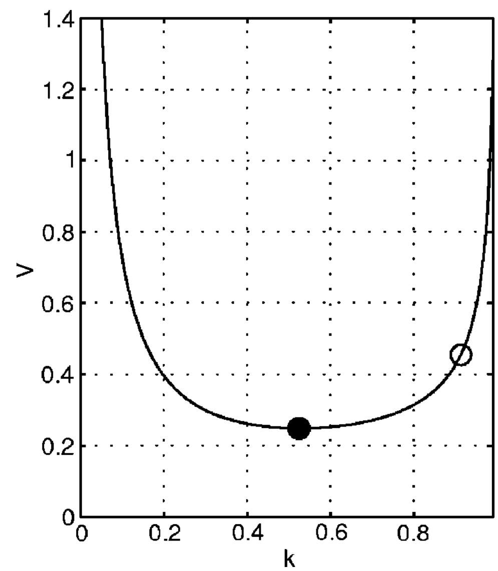

FIG. 1. A typical neutral curve [Eq. (8) for PDE's (3) with $\lambda=0$ (Le=100, $\beta=0.2$, $d=0.1667$]. Open circle denotes the threshold of stationary wave (V_o,k_o), full circle denotes the threshold of stability (V_m,k_m).

$(\sigma=0)$ (V_o,k_o). Crossing V_m in an unbounded system corresponds to an excitation of waves traveling with a finite k_m and constant speed [e.g., Fig. 3(a) before setting the control]. The critical wave number (k_m) is determined from $dV/dk=0$: $k_m^2=1-\alpha+\sqrt{\alpha^2-\alpha}$, $\alpha=\beta/d$, with a corresponding $V_m^2=(-\beta d-d^2\vartheta)/(\vartheta+\vartheta^2)$, where $\vartheta=k_m^2-1$, and group velocity of σ_m/k_m [Eq. (8)]. Note that the neutral curve possesses minimum if $\alpha>1$ $(\beta>d)$. In bounded systems, however, the boundary conditions may stabilize the front (i.e., to ensure $\sigma=0$) and crossing $V_o=\sqrt{(\beta-d^2)/(1-d)}$ will induce *stationary waves* with a corresponding wave number of $k_o=\sqrt{1-d}$ [e.g., Fig. 3(c), before setting the control]. Note that V_o (k_o) exists for $d<1$. The type of sustained patterns depends an the type of instability (absolute or convective), which can be determined analytically for unbounded system and numerically for a bounded one (see discussion and bifurcation diagram in [10]). The domain of unstable wave numbers $k_-<k<k_+$ can be determined for any $V>V_m$.

Example 1: Figure 1 presents the neutral stability curve for system (3) with Le=100, $\beta=0.2$, and $d=0.1667$. The corresponding critical wave numbers are $k_m=0.525$ $(V_m=0.245)$ and $k_o=0.9129$ $(V_o=0.45)$. For $V=0.3$, for example, the wave numbers limiting the instability in an unbounded system are $k_-=0.3$, $k_+=0.77$ (Fig. 1).

The simplest approach for control is to construct a regulator that stabilizes the unstable mode: Just above $V=V_m$, the threshold point for the appearance of waves with a finite wave number k_m (and a wavelength $T_m=2\pi/k_m$) we need a controller $\lambda=\lambda(z,t)$ with one actuator that is proportional to the unstable harmonic term with frequency k_m. Let us consider linearized PDE (6) with additive control

$$\mathrm{Le}\;\overline{x}_t + V\overline{x}_z - \overline{x}_{zz} = \overline{x} + u(\overline{x}) + \lambda. \tag{9}$$

For a sufficiently large length, which is divisible by T_m $(L/T_m=p$, integer), we approximate the deviation $\overline{x}(z,t)$ by a series of cosine eigenfunctions, using the wave number k_m that corresponds to the unstable mode in an unbounded system. With the one unstable harmonic term $x_1=c(t)\cos(k_m z)$ the temporal factor $c(t)$ is calculated via Fourier series, $c(t)=2(pT_m)^{-1}\int_0^{pT_m}\overline{x}(z,t)\cos(k_m z)dz$. Thus, the stabilizing control at $V=V_m$ becomes $\lambda=-gc(t)\cos(k_m z)$. The gain coefficient g *is* roughly evaluated as $g>\sim -k_m^2+1$ (see Appendix for details). This estimation is based on the approximate linearized spectral representation of Eq. (9) subject to no-flux boundary conditions.

For larger $V>V_m$, the range of unstable wave numbers is $k_-<k_s<k_+$. In a finite-dimension system, only a discrete set from this range corresponds to the unstable harmonics. Our purpose is to find these frequencies $\hat{k}_s$. The general approach lies in the approximation of the deviation $\overline{x}(z,t)$ in the interval $[0,L]$ by spectral methods. To this end, we use the general cosine Fourier series

$$\overline{x}(z,t) = \sum_{s=1}^{\infty} c_s(t)\cos(\hat{k}_s z + \theta_s)$$

with wave numbers $\hat{k}_s$ and a phase shift θ_s that satisfy the flux boundary conditions $[x_z(0)=Vx(0), x_z(L)=0]$. That yields

$$\hat{k}_s \tan(\hat{k}_s L) = V, \quad \theta_s = -\hat{k}_s L.$$

The analysis of the above sum reveals certain wave numbers $\hat{k}_s$ that lie in the range $[k_-,k_+]$. These terms are stabilized for a sufficiently large gain g by controller (5) with the series of l space-dependent actuators $\psi_s(z)=\cos(\hat{k}_s+\theta_s)$ and the related global-weighted sensors $\langle\overline{x}(z,t),\psi_s\rangle$, given by

$$\lambda = -g\sum_{s=s_1}^{s_l} \langle\overline{x}(z,t),\cos(\hat{k}_s z + \theta_s)\rangle\cos(\hat{k}_s z + \theta_s), \quad s_1>0. \tag{10}$$

Since k_- decreases and k_+ increases with increasing V (Fig. 1), then the number of terms in sum (10) grows with V (Fig. 2). Obviously, the number of unstable modes (l) grows with increasing length L for assigned V.

For system lengths that are divisible by wavelength ($L=pT_m$) and when V is in the vicinity V_m (the phase shift is $\theta_s\cong 0$) we can simplify control (10) as follows: $\lambda=-g\sum_{s=s_1}^{s_l}\langle\overline{x}(z,t),\cos(\hat{k}_s z)\rangle\cos(\hat{k}_s z)$, where the points $\hat{k}_s=sk_m/p\subset[k_-,k_+]$, $s>0$ (see the Appendix for derivation when $p=1$).

Example 2: For the set of parameters described in Example 1 and $L=36$, we can find $T_m=2\pi/k_m\cong 12$, $p\cong L/T_m=3$ and the set of wave numbers $\hat{k}_1=0.175$, $\hat{k}_2=0.35$, $\hat{k}_3=0.525$, $\hat{k}_4=0.7$, $\hat{k}_5=0.875,\ldots$. For $V=0.3$, we find using Fig. 1 that $k_-=0.3$, $k_+=0.77$. Thus, three wave numbers, $\hat{k}_2=0.35$, $\hat{k}_3=0.525$, $\hat{k}_4=0.7$, are enough for the above control. In the general case (i.e., L/T_m is not an integer) we need to calculate the control according to the general Eq. (10).

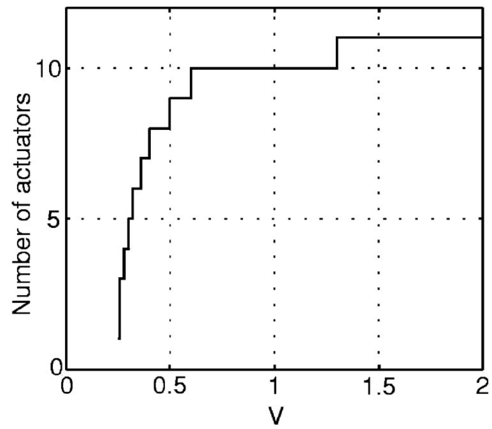

FIG. 2. The effect of V on the number of actuators required for control (10) of a finite-sized system ($L=36$) [Eq. (1) ($D=0$, $g=1.01$); other parameters as in Fig. 1].

A successful illustration of control (10), for stabilizing the homogeneous solution for various V, is shown in Fig. 3: In these and subsequent simulations we use feed boundary conditions Eq. (2c) with $x_o=0.01$ or 0.1. This deviation from steady state induces stationary or moving patterns in systems of finite size. Let us note that for large V this control does not assure a small steady state error of the spatial profile (in the convective unstable domain) [Fig. 3(d)]. Recall that this approach is approximate.

III. CONTROL BASED ON EIGENSTRUCTURE ANALYSIS (SHORT SYSTEMS)

In this section, we derive the control law based on the traditional eigenstructure analysis. The method is best suited to system where L and T_m are comparable and the influence of boundary conditions is significant.

To find the eigenfunctions, we use the spectral method to expand $\overline{x}(z,t)=x-x_s=\sum_i a_i(t)\varphi_i(z)$, where the orthonormal basis functions $\varphi_i(z)$ are the eigenfunctions of the problem: $\varphi_{izz}-V\varphi_{iz}=-\mu_i\varphi_i$, $\varphi_z|_{z=0}=V\varphi|_{z=0}$, $\varphi_z|_{z=L}=0$. Using this spectral method, we convert Eq. (9) with $\lambda(t)$ from (4) into a set of linear ODE's

$$\mathrm{Le}\ \dot{a}_j = (-\mu_j+1)a_j + \sum_i q_{ji}a_i - g\sum_{n=1}^{\eta} b_{jn}v_n(t), \quad j=1,2,\ldots \tag{11}$$

(see Appendix for derivation). Rewriting (11) in the usual vector-matrix form, we obtain

$$\mathrm{Le}\ a_t = (-\Lambda + I + Q)a + Bv, \tag{12}$$

where $a=a(t)=[a_j]$ is an infinite-dimensional vector ($j=1,2,\ldots$), v is η vector, the matrix $B=[b_{jn}]$ has infinite-dimensional columns, and $\Lambda=\mathrm{diag}(\mu_1,\mu_2,\ldots)$, $I=\mathrm{diag}(1,1,\ldots)$, $Q=[q_{ji}]$, $j,i=1,2,\ldots$ are infinite-dimensional matrices.

In the absence of control ($v=0$), the linear behavior of the homogeneous solution is described by the leading eigenval-

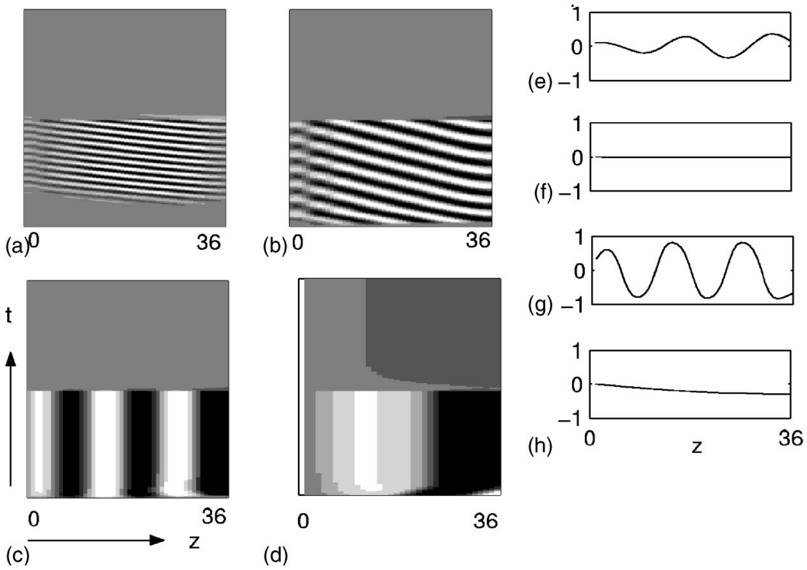

FIG. 3. Testing controller (10) by numerical simulation of Eq. (1): The figure presents gray-scale plots in the (t,z) plane of the response to a constant perturbation at the boundary conditions ($x_o=0.01$; parameters as in Fig. 2). (a) $V=0.3$, controller (10) is turned on at $t=250$ changing the traveling wave solution [the profile is presented in (e)] into a stable homogeneous solution [the profile in (f)]. (b) Same with $V=0.5$; (c) Same with $V=0.8$: the uncontrolled system exhibits stationary waves (g); (d) Same with $V=1.7$ [the final profile in (h)].

ues of matrix (12), which are typically situated in the right hand side of the complex plane for some $V_1^* < V < V_2^*$ where V_1^* denotes the stability threshold (which may be either convective or absolute), while V_2^* separates domains of absolute and convective instabilities for bounded system. For example, $V_1^*=0.26$, $V_2^*=1.88$ for the system described by the parameters $\mathrm{Le}=100$, $L=36$, $\beta=0.2$, $d=0.1667$.[1] The value $V_2^* \to V_{abs}$ as $L\to\infty$ where V_{abs} is the absolute instability threshold for unbounded PDE's (1, $D=0$), ($V_{abs}\cong 1.9$ was determined analytically in [10] for the transition from absolute to convective instability for $\mathrm{Le}=100$, $\beta=0.2$, $d=0.1667$).

The feedback control may be presented in terms of Eq. (12) as

$$v = Kw = KHa, \tag{13}$$

where the η vector $w=Ha$ is the measured output of (12) with the matrix $H=[h_{nj}]$ $(n=1,2,\ldots,\eta; j=1,2,\ldots)$. We need to find the $\eta\times\eta$ matrix K of gain coefficients such that closed-loop system (12) and (13) be asymptotically stable.

For designing the finite-dimensional control (13) we use an N-terms truncated (finite- dimensional) approximation of the dynamic matrix $A=(-\Lambda+I+Q)/\mathrm{Le}$. The truncation order N is estimated from the convergence of the leading eigenvalues of matrix A. Furthermore, the actuator functions $\psi_n(z)$ in control (4) will be set to emulate eigenfunctions of the linear operator of PDE (9) $[\psi_n(z)=\varphi_n(z)]$ and the inputs $v_n(t)=\langle x(z,t),\varphi_n^a\rangle$ correspond to the spatially weighted-average control, where φ_n^a are the adjoint eigenfunctions of linear operator of (9). The structure of matrices B and H is then simplified to $B^T=H=[I_\eta,O]$. Hence, despite the fact that the finite-dimensional control (13) with $\eta\leq N$ influences only the first diagonal $N\times N$ block of the matrix A, this control ensures stability of whole infinite-dimensional system for a sufficiently large truncation order N.

[1]For all calculations we use truncated system.

For control design, we use the approach of Gershgorin stability criterion to ensure that the truncated matrix A is diagonally dominant with negative diagonal elements [16]. Because of the dissipative nature of parabolic PDE's, only several $m<N$ first rows of the matrix A do not satisfy this stability criterion. Thus, the control[2] $v=-gI_m\, Ha$, $m\leq N$ with a sufficiently large gain g shifts the first m diagonal elements a_{ii} of A to the left part of the complex plane, so that the following inequalities hold: $a_{ii}<-\Sigma_{j=1,j\neq i}^{N}|a_{ij}|$, $i=1,\ldots,m$. As a result, the truncated matrix A (and hence the whole infinite-dimensional matrix A) becomes stable.

This control is formed in the original system [Eq. (1), $D=0$] by

$$\lambda(z,t) = -g\sum_{n=1}^{m}\langle x,\varphi_n^a\rangle\varphi_n(z). \tag{14}$$

Approaching global stability

Control (14) ensures local stability of the homogeneous solution $x=0$ of the nonlinear system via Lyapunov's linearization method. Let us demonstrate that such control, with recalculated gain coefficient g, ensures simultaneously the asymptotic stability of this solution according Lyapunov's global stability method.

We study the solution $x(z,t)$ of Eqs. (2), $[u(0)=0,\ x_0=0]$ in some neighborhood of $x_s=0$. Using the expansion $x(z,t)=\Sigma_{i=1}a_i(t)\varphi_i(z)$, where $\varphi_i(z)$ are eigenfunctions of linear operator (2a), we obtain the spectral representation of nonlinear Eq. (2a) as

$$\mathrm{Le}\ \dot{a}_j(t) = -\mu_j a_j(t) + \sum_i q_{ji}a_i(t) + f_j(a) = F_j(a), \tag{15}$$

[2]Such control is known as diagonal control (decentralized feedback control) [17].

$$j = 1,2, \dots,$$

where $a=[a_j(t)]$ is the infinite-dimensional vector ($j=1,2,\ldots$), while μ_j and q_{ji} were defined earlier [see Eq. (11)]. The nonlinear functions $f_j(a)$ are calculated as follows:

$$f_j(a) = \int_0^L (-x^3 + x)|_{x=\Sigma_i a_i(t)\phi_i(z)} \varphi_j^a(z) dz, \quad j = 1,2, \dots . \tag{16}$$

For stability analysis, we use an N-terms truncated approximation of Eq. (15), which is rewritten in the usual vector-matrix form as follows:

$$\text{Le } a_t = (-\Lambda + Q)a + f(a) = F(a), \tag{17}$$

where $a,f(a),F(a)$ are N-vectors, $Q=[q_{ji}]$ is $N\times N$ matrix and $\Lambda=\text{diag}(\mu_1,\mu_2,\ldots,\mu_N)$. Let us calculate the $N\times N$ Jacobian matrix F_a for Eq. (17) in a certain neighborhood Ω of $a=0$ [the steady state solution of (17)] defined by inequalities $|a_i|<\delta$, $\delta>0$, as

$$F_a = \frac{1}{\text{Le}} \left.\frac{\partial F}{\partial a}\right|_{|a_i|<\delta} = (-\Lambda + Q + I_N + G)/\text{Le}. \tag{18}$$

In (18), the $N\times N$ matrix $G=[G_{ij}]$ has elements $G_{ij}=-3\int_0^L(\Sigma_{n=1}^N a_n\varphi_n)^2\varphi_i\varphi_j^a dz|_{|a_i|<\delta}$, which are bounded in the region Ω. Matrix (18) is used for checking asymptotic stability of the autonomous nonlinear system based on Krasovskii's theorem [18]: the steady-state solution at the origin of nonlinear system (17) with $F(0)=0$ is asymptotically stable, according to the direct Lyapunov's method, if the matrix $F_a+F_a^T$ is negative definite[3] within some neighborhood Ω (see also contraction analysis [19,20] for nonlinear systems). The Lyapunov function for this system is $J(a)=F^T(a)F(a)$.

Let us formulate the conditions that guarantee the negative definiteness of the matrix $F_a+F_a^T$.

Assertion 1: If the matrix $F_a=[f_{ij}]$ is a diagonal dominant matrix with negative diagonal elements satisfying the inequalities

$$f_{ii} < -\sum_{j=1,j\neq i}^{N} |f_{ij}| \quad \text{and} \quad f_{ii} < -\sum_{j=1,j\neq i}^{N} |f_{ji}|, \quad i = 1, \dots, N; \tag{19}$$

then the matrix $F_a+F_a^T$ is a negative definite matrix.

Proof: If the matrix F_a has the above property, then the symmetric matrix $F_a+F_a^T$ is a stable matrix according to the Gershgorin theorem stability criteria [21]. Hence, it is the negative definite matrix.

Therefore, we need to modify the diagonal elements of the matrix F_a to satisfy the conditions of Assertion 1 for the matrix $F_a+F_a^T$. Let us introduce control $\lambda(z,t)$ (14) into nonlinear PDE's (2a). After lumping and truncating, we obtain the closed-loop nonlinear ODE's: Le $a_t=(-\Lambda+Q)a+f(a)-\text{diag}(gI_m,O)a$ that has the Jacobian matrix of $\tilde{F}_a=F_a-\text{Le}^{-1}\,\text{diag}(gI_m,O)$. Our purpose is to evaluate the gain coefficient g so that the matrix $\tilde{F}_a+\tilde{F}_a^T$ be negative definite one. Matrix F_a (18) presents the sum of the dynamics matrix $A=(-\Lambda+I+Q)/\text{Le}$ of ODE's (12) and matrix G/Le. Thus, the problem is transformed to the problem of finding the gain coefficient g, which ensures negative definiteness to the $N\times N$ matrix $(A+G/\text{Le})+(A+G/\text{Le})^T-2\,\text{Le}^{-1}\,\text{diag}(gI_m,O)$.[4] Such gain g is defined as

$$g \geqslant \min\{g_r,g_c\}, \tag{20}$$

where g_r,g_c ensure the inequalities: $f_{ii}-g_r<-\Sigma_{j=1,j\neq i}^N|f_{ij}|$, $f_{ii}-g_c<-\Sigma_{j=1,j\neq i}^N|f_{ji}|$, $i=1,\ldots,m$, where f_{ij} are elements of the matrix $F_a=A+G/\text{Le}$. Since the elements of the matrix G, are small bounded values in the region Ω, then the above conditions may be satisfied by a sufficiently large gains g_r, g_c. Thus, according to Krasovskii's theorem, the appropriate Lyapunov's function of closed-loop system is $J(a)=\tilde{F}^T(a)\tilde{F}(a)>0$.

Remark 1. The region Ω in the neighborhood of the equilibrium point $a=O$ where the matrix $\tilde{F}_a+\tilde{F}_a^T$ is negative definite defines the domain of attraction of the stable homogeneous solution. The number of terms (m) and the value of gain coefficient (g) in control (14) affects the size of the domain of attraction of the closed-loop system: this domain expands with increasing m and g.

To demonstrate these results, we apply control strategy (14) to system (1) with $L=10$ ($D=0,\beta=0.45,d=0.2$) and various V. The homogeneous solution of such a system is unstable for $0.8\leqslant V\leqslant 1.5$. Analysis of the dynamics matrix A of ODE's (12) reveals that the first three rows do not satisfy the Gershgorin *stability criterion* for V in the above range. Analysis of the diagonal elements of the dynamics matrix A shows that control (14) with $m=2$, $g=0.48$ is sufficient to stabilize the small deviations of the homogeneous solution for $V\geqslant 0.8$. Here, $\varphi_i(z)\sim e^{0.5Vz}[\cos(\sigma_i z)+0.5V\sin(\sigma_i z)/\sigma_i]$, $\varphi_j^a(z)=\varphi_j(z)e^{-Vz}$, $i=1,2$, and $\sigma_1,\sigma_2(\sigma_1\neq 0)$ are the smallest roots of $\tan(\sigma_i L)=V\sigma_i/(\sigma_i^2-0.25V^2)$. The number of terms in control (14) grows with increasing system length (eg., $m=5$ for $L=20$, $V\geqslant 0.8$, other parameters $D=0$, $\beta=0.45$, $d=0.2$). Control (14) effectively suppresses patterns in absolutely or in convectively unstable domains (Fig. 4).

The number of terms in control (14) depends on the number of positive eigenvalues of matrix (12). That number passes through a maximum with growing V in the range $[V_1^*,V_2^*]$, where V_1^* (the stability threshold) and V_2^* (the absolute-convective threshold for bounded system) were defined before (e.g., for $L=20$, $\text{Le}=100$, $D=0$, $\beta=0.45$, $d=0.2$, we find $V_1^*=0.8$, $V_2^*=1.5$, and the maximal number of terms occurs at $\hat{V}=1.2$). For $V>V_2^*$, matrix (12) has only negative eigenvalues and control is not required for asymptotic stability. However, the constant perturbation imposed at the boundary [condition (2c), with $x_o=0.01$ in our

[3]A necessary and sufficient condition for a symmetric matrix to be negative definite is that its odd order principal minors are negative and even order ones are positive or, equivalently, that all its eigenvalues be strictly negative.

[4]m is an upper bound on the number of first rows (columns) of A that do not satisfy the Gershgorin stablility criterion.

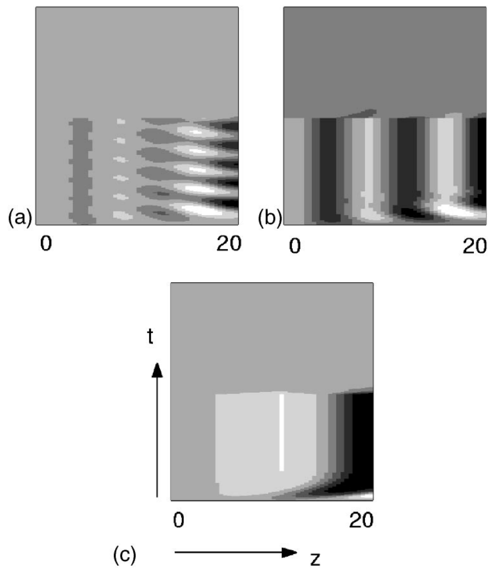

FIG. 4. Testing controller (14) by numerical simulation of Eq. (1) [$D=0$, $\mathrm{Le}=100$, $\beta=0.45$, $d=0.2$, $L=20$]: The figure presents gray-scale plots in the (t,z) plane of the response to constant perturbation at the boundary conditions. (a) $V=0.8$, $x_o=0.01$, controller (14) is turned on at $t=250$, changing the traveling wave solution, into a stable homogeneous solution, (b) Same with $V=0.8$, $x_o=0.1$. (c) Same with $V=2.1$, $x_o=0.01$.

case] induces stationary patterns in the system because in this range (convective instability) the stationary wave solution has a large attraction domain. Recall also that we search for a control that will assure a sufficiently wide domain of stability. Consequently, we find that the number of terms in control does not decrease for $V>\hat{V}$.

The above control is effective only for short systems. For the case studied in Fig. 5, the weighted global control (14) requires six actuators/sensors [Fig. 5(a)], compared with approximate controller (10), which requires three actuators

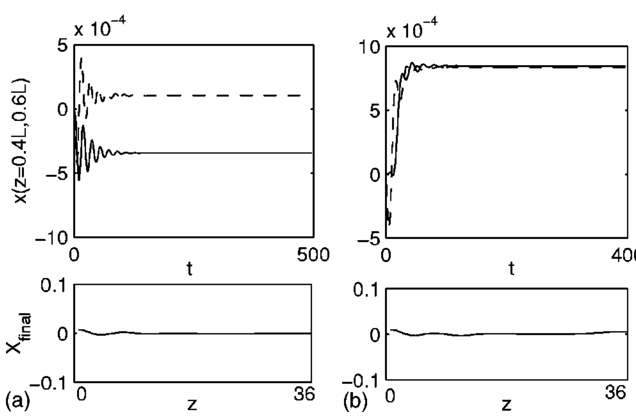

FIG. 5. Comparisons of controller design by exact (eigenstructure analysis, Eq. (14)) method [(a) $m=6$, $\sigma_1=0.063$, $\sigma_2=0.206$, $\sigma_3=0.286$, $\sigma_4=0.368$, $\sigma_5=0.452$, $\sigma_6=0.537$] with approximate design [(b), controller (10) with 3 actuators: $\hat{k}_2=0.45,\hat{k}_3=0.536,\hat{k}_4=0.62$]. The figure presents the temporal response at $z=0.4L,0.6L$ (upper row) and the stabilized spatial profile (second row) ($V=0.27$, $L=36$, other parameters as in Fig. 1. $x_o=0.01$).

[Fig. 5(b)]. However, the former control ensures a smaller steady-state error.

IV. PINNING CONTROL

By pinning control, we imply control that influences the state variable at a finite number of discrete points $z_i\subset[0,L]$. To test it, we set $x(z_i,t)=x_s(z_i)$, where x_s is the homogeneous solution. We need to determine the spacing between pinning positions required to assure stability of the system. Consider a control that pins the homogeneous $x=0$ solution at two points z_1 and z_2, so that

$$x(z_1)=0,\quad x(z_2)=0. \tag{21}$$

The ability of this controller to stabilize the system can be checked now by replacing the original problem (3) with the following three problems:

$$\mathrm{Le}\, x_t+Vx_z-x_{zz}=-x^3+x+u(x),\quad x_z\big|_{z=0}=Vx\big|_{z=0},\quad x\big|_{z_1}=0, \tag{22a}$$

$$\mathrm{Le}\, x_t+Vx_z-x_{zz}=-x^3+x+u(x),\quad x\big|_{z_1}=0,\quad x\big|_{z_2}=0, \tag{22b}$$

$$\mathrm{Le}\, x_t+Vx_z-x_{zz}=-x^3+x+u(x),\quad x\big|_{z_2}=0,\quad x_z\big|_{z=L}=0, \tag{22c}$$

that describe the system near the left boundary, between two intermediate pins and near the right boundary, respectively.

Linearization of (22a)–(22c) around $x=0$ and approximation of the linearized system by spectral methods leads to the following linear systems in the three domains

$$\mathrm{Le}\, a_t=Aa,\quad A_{ij}=(-\mu_j+1)\delta_{ij}+\langle u(\varphi_i),\varphi_j^a\rangle\quad i,j=1,2,\ldots, \tag{23}$$

where $\delta_{ij}=1$ when $i=j$ and $\delta_{ij}=0$ otherwise, and $\varphi_i(z)$ $[\varphi_j^a(z)]$ are the eigenfunctions (adjoint eigenfunctions) of the linear operator (22) subject to corresponding boundary conditions (Eqs. (22a)–(22c), while μ_j are the related eigenvalues. For these domains (22a)–(22c) these are, respectively,

$$\mu_j=\sigma_j^2+0.25V^2,$$

$$\varphi_i(z)=\Theta_i e^{0.5Vz}\left[\cos(\sigma_i z)+\frac{V}{2\sigma_i}\sin(\sigma_i z)\right], \tag{24a}$$

$$2\sigma_j=-V\tan(\sigma_j z_1),$$

$$\mu_j=(j\pi)^2/(z_2-z_1)^2+V^2/4,$$

$$\varphi_i(z)=\Theta_i e^{V(z-z_1)/2}\sin[i\pi(z-z_1)/(z_2-z_1)], \tag{24b}$$

$$\mu_j=\sigma_j^2+0.25V^2,$$

$$\varphi_i(z)=\Theta_i e^{0.5Vz}[\sin(\sigma_i z)-\tan(\sigma_i z_2)\cos(\sigma_i z)], \tag{24c}$$

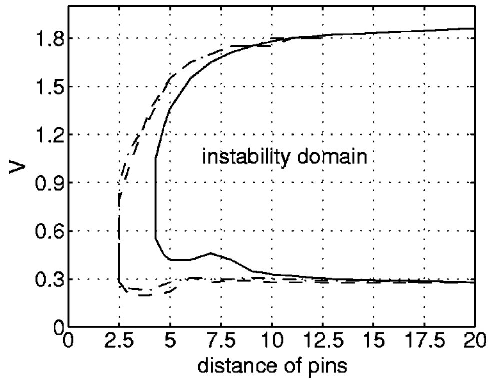

FIG. 6. Analysis of stability domains of pinning control: Figure shows the effect of V on pin separation on the left (z_1, dashed line), middle (z_2-z_1, solid line), and right ($L-z_2$, dashed-dotted line) domains. The stability was determined from Eqs. (23) (parameters are as in Fig. 1).

$$2\sigma_j = -V\tan[\sigma_j(L-z_2)].$$

In (24), $\Theta_i = \langle \varphi_i, \varphi_i^a \rangle^{-0.5}$. {The eigenfunctions for case (24a) were derived in [22]}.

For boundary conditions (22b), we find the exact expression for $\langle u(\varphi_i), \varphi_j^a \rangle$ as

$$\langle u(\varphi_i), \varphi_j^a \rangle = -\frac{2\beta}{V\tilde{L}(\tilde{V}^2 + i^2\pi^2/\tilde{L}^2)} \left\{ \frac{ij\pi^2}{\tilde{L}^2(\tilde{V}^2 + j^2\pi^2/\tilde{L}^2)} + \vartheta \right\}, \tag{25}$$

where $\tilde{L}=z_2-z_1$, $\tilde{V}=d/V+V/2$, $\vartheta=0.5\tilde{V}\tilde{L}$ for $i=j$, $\vartheta=2i^2/(j^2-i^2)$ for $i\neq j$ (odd $i+j$), $\vartheta=0$ for $i\neq j$ (even $i+j$).

Linear stability analysis of Eq. (23), using eigenstructure (24), enables us to calculate the maximal pin separations (z_1, $L-z_2$. and z_2-z_1) that assure a homogeneous stable solution for various V (Fig. 6): All these separations tends to infinity at V_1^*, V_2^* (the stability and absolute-convective instability thresholds). Thus, a single pin is sufficient for stabilizing system of any length in a domain of convective instability ($V>V_2^*$).

Let us note that pinning control requires control of the state variable at certain points, which is typically technologically difficult. Moreover, unlike the feedback controllers considered above (Secs. II and III), pinning control ensures local stability for small perturbations of initial or boundary conditions. However, pinning control may require fewer terms than control based on spectral representation.

V. PINNING BOUNDARY CONDITIONS

The results of previous section suggest that pinning at the feed boundary may be most effective in convective-unstable systems. Consider a special type of pinning control in which we control the feed conditions to obey $x(0)=0$ at all times. With this control, the original problem is replaced by the right-boundary analysis above [Eq. (22c), $z_2=0$] and linear stability analysis (Fig. 6) and simulations [Fig. 7(a)] verify that in a domain of absolute instability, this control stabilizes

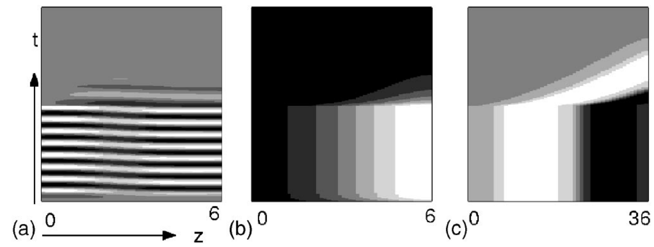

FIG. 7. Testing of pinning boundary control [Eq. (1), $D=0$] in a short system ($L=6$) with $V=0.3$ (a) and $V=1.7$ (b) and in a long system ($L=36, V=2.1$) (c). Other parameters as in Fig. 1; The figure presents gray-scale plots in the (t,z) plane of the response to constant perturbation at the boundary conditions ($x_o=0.01$), and pinning control [i.e., $x(0)=0$] is turned on at $t=250$.

the homogeneous solution in a short system, while in a domain of convective instability ($V=2.1$) this control is effective for a system of any size [Figs. 7(b) and 7(c)].

VI. APPLICATION TO THE CROSS-FLOW REACTOR

We try to apply the methodology above to control a homogeneous solution of the homogeneous model of a cross-flow reactor in which the feed is evenly distributed along the reactor (e.g., through a membrane) and the reactor is evenly cooled (see [23] for a detailed description). By a homogeneous model, we imply that interphase gradients of temperature and concentration are absent. The model for a first-order Arrhenius kinetics, $r=A\theta e^{-E/RT}C_A$, with mass supply and heat removal along the reactor is written below in dimensionless form using conventional notation for variables and parameters

$$\mathrm{Le}\frac{\partial y}{\partial \tau} + \frac{\partial y}{\partial \xi} - \frac{1}{\mathrm{Pe}}\frac{\partial^2 y}{\partial \xi^2} = B\,\mathrm{Da}(1-x)\theta(y) + \beta_T(Y-y) + \lambda$$
$$= f_1(x,y) + \lambda, \tag{26a}$$

$$\frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial \xi} = \mathrm{Da}(1-x)\theta(y) + \beta_C(X-x) = f_2(x,y),$$
$$\theta(y) = \exp\left(\frac{\gamma y}{\gamma + y}\right), \tag{26b}$$

where x and y, respectively, are dimensionless concentration and temperature, Pe and Da are Peclet and Damkohler numbers, while X and Y denote wall concentration and temperature, respectively, λ is the control action. We assume negligible dispersion of mass and typically the bed heat capacity is large (Le≫1) and conductivity is small, Pe≫1. The Danckwert's boundary conditions are typically imposed on the model:

$$y_\xi|_{\xi=0} = \mathrm{Pe}[y|_{\xi=0} - y_{in}]; \quad y_\xi|_{\xi=L} = 0; \quad x|_{\xi=0} = x_{in}. \tag{26c}$$

In the absence of control, the right-hand side of Eq. (26) $[f_1(x_s,y_s)=f_2(x_s,y_s)=0]$ may admit a unique or multiple solutions for a certain range of wall transport parameters β_C

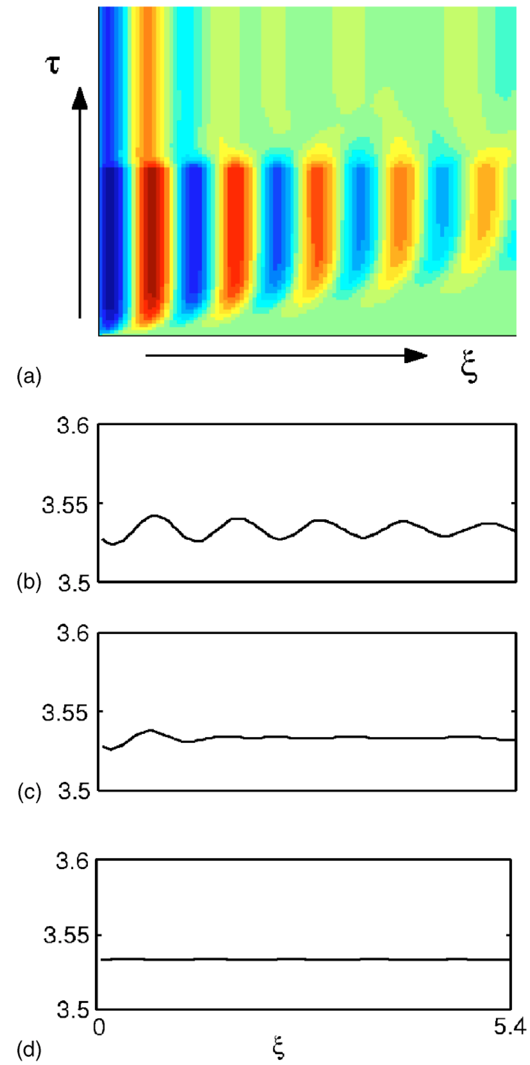

FIG. 8. (Color online) Simulation and control of the cross-flow reactor of length $L=5.4$ [Eq. (26), Le=100, Pe=15, $B=16.2$, $\beta_T=4$, $\beta_c=1$, Da=0.2, $X=0$, $Y=0$, $\gamma=10^4$]. (a) Response of the closed-loop system with control (10) to constant small perturbation at the boundary conditions ($y_{in}=3.53$, $x_{in}=0.872$ compared with steady-state solution $y_s=3.5334$, $x_s=0.8724$). The figure presents gray-scale plots in the (τ,ξ) plane. Controller (10) using four actuators is turned on at $t=250$, changing the uncontrolled stationary wave solution (b) into a homogeneous solution (c). Pinning the inlet conditions $y_{in}\cong y_s=3.533$, $x_{in}=x_s$ assures the homogeneous solution exactly (d).

and β_T. We limit this analysis to the case of a unique unstable solution that leads to stationary or moving patterns. The neutral curve that describes the stability boundary relation between Pe(k) and wave number (k) was derived in [23], and we use here the analysis and results presented there.

For the case simulated in Fig. 8, the homogeneous steady-state value is $y_s=3.5334$, $x_s=0.8724$. The system is convectively unstable [23] and fixing $y_{in}\cong y_s=3.533$, $x_{in}=x_s$ assures a homogeneous solution [Fig. 8(d)]. Small perturbations from these values lead to stationary patterns [Fig. 8(b), the uncontrolled case]. We attempt to apply control in that case. As we show, control can be useful for small deviations but will not be useful for large ones.

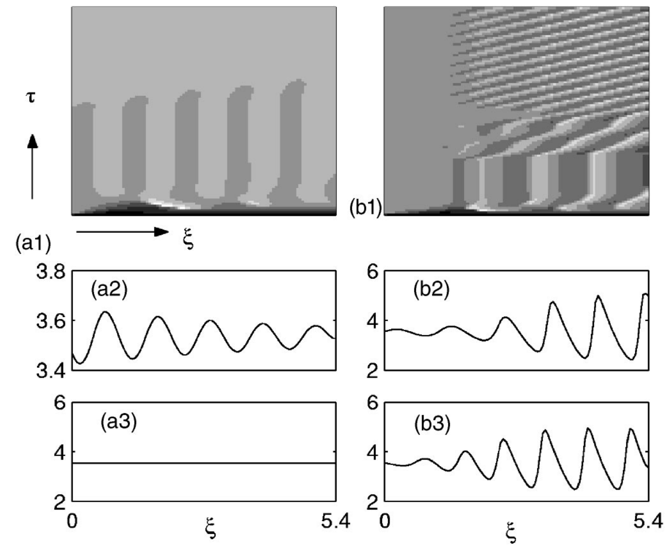

FIG. 9. Testing of pinning boundary control with uncertainty of parameters by setting $y_{in}=L^{-1}\int_0^L y(\xi,\tau)d\tau$, $x_{in}=x_s=0.872\,44$ in system (26) of length $L=5.4$ and Pe=15 (a) and Pe=25 (b); other parameters as in Fig. 8. Pinning control is turned on at $t=250$, changing the stationary pattern solution (a) into a stable homogeneous one for Pe=15 or into moving waves (b) for Pe=25. For $t<250$, $y_{in}=3.5$ (compared with $y_s=3.5334$). Figures (a1,b1) presents gray-scale plots while (a2,b2) and (a3,b3) present spatial profiles before and after control.

The simplest approach to control is to capitalize on the convective instability by pinning the boundary at the value of the homogeneous solution (Sec. V). In the general case the homogeneous solution is not known exactly, and the solution is very sensitive to this value. We can estimate it from integrating the variables x and y over a certain domain in the reactor ($\xi^*<\xi<L$), as

$$y_{in}(t)=\frac{1}{L-\xi^*}\int_{\xi^*}^{L}y(\xi,t)d\xi;\quad x_{in}(t)=\frac{1}{L-\xi^*}\int_{\xi^*}^{L}x(\xi,t)d\xi. \tag{27}$$

Obviously, $y_{in}\to y_s$ (similarly for x) for small amplitude spatially harmonic patterns and when $L-\xi^*$ is an integer multiple of the wavelength of the solution. We have attempted this approach numerically (Fig. 9) and it worked well with $x_{in}=x_s$ and for small deviations from the critical point [Pe=15, Fig. 9(a)], but with larger deviations [Pe=25, Fig. 9(b)] it converted the stationary patterns into moving waves. This may be due to residual error of integrating a fractional number of wavelengths and can be corrected by integration in time as well. A formal analysis of the stability of this approach will be conducted elsewhere.

We test now the traditional space-dependent actuator. To choose the methodology of control, we approximate the present problem in the form similar to Eq. (2). For Le$\gg$1, we can ignore $\partial\xi/\partial\tau$ terms. Defining

$$\phi==(Y-y)-B(X-x),$$

we find from Eqs. (26) the ODE

$$\phi_\xi + \beta_c \phi = -\mathrm{Pe}^{-1} y_{\xi\xi} - (\beta_c - \beta_T)(y - Y)$$

with feed condition $\phi(0)=\phi|_{\xi=0}=Y-y(0)-B[X-x(0)]$, where $y(0)=y|_{\xi=0}$, $x(0)=x|_{\xi=0}=x_{in}$. After some rearrangement (similar to one used in [24]), we obtain the following nonlinear PDE with respect to y, as

$$\mathrm{Le}\frac{\partial y}{\partial \tau} + \frac{\partial y}{\partial \xi} - \frac{1}{\mathrm{Pe}}\frac{\partial^2 y}{\partial \xi^2} = F(y) + u(y) - \mathrm{Da}\ \Psi(\xi)\theta(y), \tag{28a}$$

$$F(y) = \mathrm{Da}[B(1+X) - (1+\beta)(Y+y)]\theta(y) - \beta_T(y-Y),$$
$$\beta = \beta_T - \beta_C, \tag{28b}$$

$$u(y) = -\mathrm{Da}\ \theta(y)\beta\left[\int_o^\xi e^{\rho-\xi} y d\rho - y\right], \tag{28c}$$

$$\Psi(\xi) = e^{-\beta_C \xi}\{Y - y(0) + B[x(0) - X]\} + [y(0) - \beta Y]e^{-\xi}. \tag{28d}$$

We can approximately convert Eq. (28) to the model studied here if we set $\Psi(\xi)=0$, either because $x(0)=0$, $y(0)=0$ and/or since the effect of initial conditions decayed ($\xi \gg 1$), and approximate the exponential term $\theta(y)=\exp[\gamma y/(1+\gamma y)]$ as $(1+by^2)$. The structure of Eq. (28a) is similar to that of Eq. (2) after replacing the space coordinate by $z=\xi\sqrt{\mathrm{Pe}}$. Thus, we may use the control action $\lambda=\lambda(\xi,\tau)$ of the general form (4) for stabilization of the small deviations from the homogeneous state y_s of Eq. (26).

We now apply the methodology of Sec. II to demonstrate reasonable results for control of short system with small perturbations of feed conditions from the homogeneous solution. Here, control (10) is constructed using the wave numbers $\hat{k}_s$ (and phase shifts θ_s) that satisfy the flux boundary conditions and lie in the domain of unstable wave numbers. That domain for a bounded system (26), $[\tilde{k}_-, \tilde{k}_+]$, may differ from that calculated from neutral curve for unbounded system (26), $[k_-, k_+]$. For example, for the set of parameters in Fig. 8, we calculated from the neutral curve {Fig. 1(a), [23]} the following critical points: $k_m=6.68$ ($\mathrm{Pe}_m=14.976$), $k_o=5.93$ ($\mathrm{Pe}_o=15.42$); for $\mathrm{Pe}=15$, the wave numbers $k_-=6.4$, $k_+=6.95$ limit the instability domain in an unbounded system. Following the approach in Sec. II, we should use two actuator/sensors with wave numbers 6.61 and 6.93 that employ eigenfunctions that are within this unstable domain. Numerical simulations of Eq. (26) revealed that this may not be sufficient. We added two actuators with adjacent wave numbers (5.46, 6.03) and found that control (10) with four actuator/sensors ($\hat{k}_1=5.46$, $\hat{k}_2=6.03$, $\hat{k}_3=6.61$, $\hat{k}_4=6.93$) is sufficient for stabilization of the homogeneous solution of system of length $L=5.4$ (Fig. 8).

This analysis suggest that the approaches presented here are very limited in their applications for real systems. We should empahsize again that control of convectively unstable systems is different than that traditionally applied to distributed-parameters systems and should capitalize on pinning the feed conditions.

VII. CONCLUDING REMARKS

We study the problem of stabilization of a homogeneous solution in two-variable reaction-convection-diffusion system with oscillatory kinetics. We choose a model that incorporates convection since the interplay between absolute instability due to autocatalytic kinetics and convective stability, that can be induced by convection, makes the control problem especially interesting. While spatially weighted feedback control can suppress patterns in absolutely or convectively unstable systems, the number of required actuators increases dramatically with system size and with the distance from bifurcation point. Patterns can also be suppressed by pinning the desired solution at several set points of the actual system size. A control based on pinning of the boundary condition is most simple and effective in convectively unstable systems, if the system parameters are known exactly. For systems with a certain uncertainty of the parameters, we introduce a feedback control of feed conditions. Implementation of these approaches in a more realistic model of a cross-flow reactor highlights the difficulties associated with controlling the homogeneous system in distributed systems.

While we have considered here only boundary conditions that are constant in time, we should be aware that this is a particular case of the problem with time-dependent periodic or noisy boundary condition. That may introduce families of traveling wave solutions, which may interact to form spatially and periodically chaotic solutions. This issue will be addressed in future presentations.

ACKNOWLEDGMENTS

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APPENDIX

1. Derivation of control at $V=V_m$

For simplification, we consider the case $L=T_m$. Let us expand $\overline{x}(z,t)$ on interval $[0,L]$ in cosine Fourier series

$$\overline{x}(z,t) \cong 0.5\sqrt{L}\sum_{n=1}^{\infty} c_n(t)\varphi_n(z), \tag{A1}$$

where $\varphi_1=1/\sqrt{T_m}$ and $\varphi_n(z)=\sqrt{2/T_m}\cos[2\pi(n-1)z/T_m]$, $n=2,3,\ldots$. Members of the series above are orthonormal in the interval $[0,L]$ and we approximate the linearized system (9) with $\lambda=0$ by a set of ODE's

$$\mathrm{Le}\ \dot{c}_1 = c_1 - (\beta/V)\sum_{i=1}^{\infty} q_{1i}c_i, \tag{A2}$$

$$\mathrm{Le}\ \dot{c}_n = \{-[(n-1)k_m]^2 + 1\}c_n + V\sum_{i=1}^{\infty} p_{ni}c_i - (\beta/V)\sum_{i=1}^{\infty} q_{ni}c_i, \quad n=2,3,\ldots, \tag{A3}$$

where elements $q_{ni}=\langle u(\overline{x}), \varphi_n(z)\rangle$ and $p_{ni}=\langle V\overline{x}_z, \varphi_n(z)\rangle$ are calculated by integrating the term $u(\overline{x})$ [Eq. (2b), $u(0)=0$] and $V\overline{x_z}$ with weight functions. By exact integration, we obtain the coefficients in Eq. (A2) as

$$q_{11} = \theta[\theta e^{-L/\theta}/L - (\theta/L - 1)], \quad \theta = 2V/d,$$

$$q_{1i} = -\frac{4}{L}\left[\frac{e^{-L/\theta}}{(d/V)^2 + (i-1)^2 k_m^2}\right], \quad i = 2,4,\ldots, \quad q_{1i} = 0, \quad i = 3,5,\ldots, \tag{A4}$$

which for $L \gg 1$ becomes $q_{11} \cong \theta$, $q_{1i} \cong 0$, $i > 1$. Thus, the variable $c_1(t)$ is stable for any V and $\beta < d$, $d < 1$. This is also evident from the neutral curve.

Stability analysis of the structure of Eqs. (A3) with parameters Le=100, $\beta=0.2$, $d=0.1667$ (and $k_m=0.525$) shows that the variables $c_n(t)(n>2)$ are stable for any V, while the variable $c_2(t)$ is unstable for $V > V_m$. Since $\varphi_2(z) \sim \cos(k_m z)$, then the system may be stabilized by the control of the simplest form $\lambda = -g\langle \overline{x}, \cos(k_m z)\rangle \cos(k_m z)$, with a sufficiently large gain g. A rough estimate of gain $(g > -k_m^2 + 1)$ follows from Eq. (A3) $(n=2)$.

2. Derivation of Eq. (11)

Substituting $\overline{x}(z,t)_s = \Sigma_i a_i(t)\varphi_i(z)$ with $\lambda(t)$ from (4) and integrating with a weight eigenfunctions $\varphi_j^a(z)$ we convert Eq. (9) into a set of linear ODE's

$$\mathrm{Le}\ \dot{a}_j = (-\mu_j + 1)a_j + \sum_i q_{ji}a_i - g\int_0^L \sum_{n=1}^{\eta} v_n(t)\psi_n(z)\phi_j^a(z)dz, \tag{A5}$$

where $\mu_j = \sigma_i^2 + 0.25V^2$, $\varphi_i(z)$, and $\varphi_j^a(z)$ are eigenvalues, eigenfunctions, and adjoint eigenfunctions of linear operator (3) with flux boundary conditions, and $\sigma_j > 0$, $i = 1,2,\ldots$ satisfy the transcendental equation (see [22] for concrete formulas), $q_{ji} = \langle u(\varphi_i)\varphi_j^a\rangle$, where $u(\varphi_i)$ satisfies Eq. (2b) with $x = \varphi_i$ and $u(0)=0$. Let us evaluate the last term in (A5). Since

$$\int_0^L \sum_{n=1}^{\eta} v_n(t)\psi_n(z)\phi_j^a(z)dz = \sum_{n=1}^{\eta} v_n(t)\int_0^L \psi_n(z)\phi_j^a(z)dz,$$

then denoting

$$b_{jn} = \int_0^L \psi_n(z)\phi_j^a(z)dz, \tag{A6}$$

we can finally rewrite (A5) as Eq. (11).

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